The Teleparallel Lagrangian and Hamilton-Jacobi Formalism

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Abstract

We analyze the Teleparallel Equivalent of General Relativity (TEGR) from the point of view of Hamilton-Jacobi approach for singular systems.
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1 Introduction

The analysis of singular systems is an interesting problem in Physics, as such systems appear in many relevant physical problems. Such analysis is usually carried out using the generalized Hamiltonian formulation, developed by Dirac [1, 2, 3, 4], where the canonical Hamiltonian is not uniquely determined due to the singularity of the Hessian matrix; what causes the appearance of relations between canonical variables. These constraints, multiplied by Lagrange multipliers, are added to the canonical Hamiltonian and consistency conditions are implemented to eliminate some degrees of freedom of the system.

Despite the outstanding success of Dirac's formalism, new approaches to the analysis of singular systems are always welcome because they may reveal new mathematical and physical information about the system in study. Among others, an alternative method to analyze singular systems is the Hamilton-Jacobi formalism [5, 6], which has been used in many examples [7, 8, 9] and generalized to higher order singular systems [10, 11] and systems with Berezinian variables [12]. This formalism uses Carathèodory's equivalent Lagrangians method [13] to write a set of Hamilton-Jacobi partial differential equations from which one can obtain the equations of motion as a set of total differential equations in many variables.

One example of physical system described by a singular Lagrangian, and that has already been studied through Dirac's method [14, 15, 16], is the Teleparallel Equivalent of General Relativity (TEGR) which is an alternative formulation of General Relativity [17] developed in Weitzenböck space-time [18]. In oposition to General Relativity, in TEGR the curvature tensor vanishes but the torsion tensor does not so that, in this geometrical framework, the gravitational effects are caused by the torsion tensor and not by curvature.

The TEGR has been successfully analyzed through Dirac's Hamiltonian formalism [14, 15] and generated successfull applications [19]. Moreover, many of the characteristics regarding the interaction of spin 0, 1 and spinor fields in TEGR have been studied recently [20, 21, 22, 23, 24] as well as its gauge symmetries [25]. Others aspects of TEGR, as the energy-momentum tensor and geodesics and "force" equation are addressed in reference [26] and in the references cited therein.

Our intention in this work is to add a different point of view to the analysis of TEGR by studying it through the above mentioned Hamilton-Jacobi formalism for singular sistems. First, we will introduce the Lagrangian density of TEGR in a form which is appropriate to our approach. Then we address the basic aspects of Hamilton-Jacobi formalism for singular systems and apply such formalism to TEGR. Finally, we present our final comments.

2 The Lagrangian of TEGR

In this section we summarize the Lagrangian formulation of TEGR in terms of the tetrad field, as presented in reference [15] where a global $SO(3,1)$ symmetry is taken from outset. This choice is done because, when starting from a local $SO(3,1)$ symmetry, it may not be possible, with certainty reference systems choices, to obtain a set of first class constraints [14].

So, we take the Lagrangian density of TEGR in empty space-time given, in terms of the tetrad field $e_{a\mu}$, by

$$L(e) = -k\, e\, \Sigma^{abc} T_{abc}, \tag{1}$$

where Latin letters are $SO(3,1)$ indexes (taking values $(0), ...(3)$), Greek letters are space-time ones (taking values $0, ..., 3$), $e = det(e^a{}_\mu)$, $k = \frac{1}{16\pi G}$ and G is the gravitational

constant. Besides that, the torsion tensor $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ is defined in terms of the tetrad field as

$$T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu},$$

and its trace is defined as

$$T_b = T^a{}_{ab};$$

while the tensor Σ^{abc} is defined as

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c),$$

such that

$$\Sigma^{abc}T_{abc} = \frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a.$$

The fields equations can be obtained from the variation of L with respect to $e^{a\mu}$ and are equivalent to Einstein's equations in tetrad form [14]

$$\frac{\delta L}{\delta e^{a\mu}} \equiv \frac{1}{2}\, e\left\{R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\right\}. \tag{2}$$

Let us now consider the tetrad field $e^{a\mu}$ in terms of the $3+1$ decomposition. In this case the space-time manifold is assumed to be topologically equivalent to $M \times R$, where M is a noncompact three-dimensional manifold. We consider that ${}^4e_{a\mu}$ is a tetrad field for $M \times R$. In terms of the lapso N and shift N^i $(i, j, k... = 1, ..., 3)$ functions we have

$${}^4e^a{}_i = e^a{}_i,$$

$${}^4e^{ai} = e^{ai} + (N^i/N)\eta^a,$$

$${}^4e^a{}_0 = e^a{}_iN^i + \eta^aN,$$

$$e^{ai} = e^a{}_k\bar{g}^{ki},$$

$$\eta^a = -N\,{}^4e^{a0},$$

where e_{ai} and e^{ai} are now restricted to M. Moreover, η^a is an unit timelike vector such that $\eta_a e^a{}_i = 0$ and $\bar{g}^{ik}$ is the inverse of $g_{ik} = e_{ai}e^a{}_k$. The determinant of the tetrad field is now given by ${}^4e = Ne$, where $e = det(e^{(l)}{}_k)$. In terms of the 3+1 decomposition, the TEGR Lagrangian density L can be written as

$$L = \frac{ke}{2N}\left(\bar{g}^{ik}l^{(l)}_i l_{(l)k} + e^{(0)i}e^{(0)k}l^{(l)}{}_i l_{(l)k} + e^{(l)i}e^{(n)k}l_{(n)i}l_{(l)k} - 2e^{(l)i}e^{(n)k}l_{(l)i}l_{(n)k}\right)$$

$$-4ke\bar{\Sigma}^{a(0)i}l_{ai} - kNe\bar{\Sigma}^{abc}\bar{T}_{abc}$$

where we made the use of the following definitions

$$l_{ai} = \dot{e}_{ai} - \eta_a\partial_i N - e_{aj}\partial_i N^j - N^j\partial_j e_{ai},$$

$$\bar{\Sigma}^{abc} = \frac{1}{4}\left(\bar{T}^{abc} + \bar{T}^{bac} - \bar{T}^{cab}\right) + \frac{1}{2}\left(\eta^{ac}\bar{T}^b - \eta^{ab}\bar{T}^c\right),$$

$$\bar{T}^{abc} = e^{bi}e^{cj}T^a{}_{ij}.$$

The momenta Π^{ak} conjugated to e_{ak} are obtained by

$$\Pi^{ak} = \frac{\delta L}{\delta \dot{e}_{ak}},$$

where

$$\Pi^{(0)k} = 4ke\bar{\Sigma}^{(0)(0)k},$$

end

$$\Pi^{(r)s} = -\frac{ke}{N}\left(l^{(r)s} + e^{(0)i}e^{(0)s}l^{(r)}{}_i + e^{(r)i}e^{(n)s}l_{(n)i} - 2le^{(r)s}\right) + 4ke\bar{\Sigma}^{(r)(0)s}.$$

Note that here $l^{(l)i} = \bar{g}^{ik}l^{(l)}{}_k$ and $l = e^{(l)i}l_{(l)i}$.

In order to simplify the calculations we can make a choice of reference frame, analogous to the one made in ADM formulation. This choice is usually refered in literature as the Schwinger's time gauge [28], $e_{(0)i} = e_{(0)}{}^i = 0$, so that the momenta defined above are

such that $\Pi^{ks} = \Pi^{sk}$ and $\Pi^{(0)k} = \bar{T}^{(l)}{}_{(l)}{}^{k}$. In this case it is possible to write the Lagrangian density L as [14]

$$L = \Pi^{(l)i}\dot{e}_{(l)i} - H_c, \tag{3}$$

where

$$H_c = NC + N^i C_i, \tag{4}$$

$$C = \frac{1}{4e}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right) + e\Sigma^{ijk}T_{ijk} - 2\partial_i(eT^i), \tag{5}$$

and

$$C_i = e_{(l)i}\partial_k\Pi^{(l)k} + \Pi^{(l)k}T_{(l)ki}. \tag{6}$$

In Eq. (3) there is no time derivatives of the functions N and N^i, so the Lagrangian density L is singular. With these results, we now can investigate the integrability conditions of the Lagrangian density of the TERG, given in Eq. (3), in the Hamilton-Jacobi formalism, what it will be done in the next section.

3 The Hamilton-Jacobi Formalism

The Hamilton-Jacobi formalism, recently developed to analyze singular systems [5, 6, 10, 12], uses the equivalent Lagrangians method [13] to obtain a set of Hamilton-Jacobi partial differential equations [5, 6]. We suggest the references just mentioned for details and present here only the main aspects of this formalism. For this, let us consider a singular Lagrangian function $L = L(q_i, \dot{q}_i, t)$, where $i = 1, ..., N$. The Hessian matrix is then given by

$$H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad i, j = 1, ..., N. \tag{7}$$

Being the rank of the Hessian matrix $P = N - R < N$, we can define, without loss of generalization, the order of the variables q_i in a such way that the $P \times P$ matrix in the

right bottom corner of the Hessian matrix be nonsingular. So there will be R relations among canonical variables given by

$$p_\alpha = -H_\alpha \left(q^i;\ p_a\right);\ \alpha = 1, ..., R; \tag{8}$$

which correspond to Dirac's primary constraints $\Phi_\alpha \equiv p_\alpha + H_\alpha \left(q^i;\ p_a\right) \approx 0$. From these we get the Hamilton-Jacobi partial differential equations, given by

$$H'_\alpha = H_\alpha(t, q_i, p_a) + p_\alpha = 0, \tag{9}$$

$$H'_0 = H_c(t, q_i, p_a) + p_0 = 0, \tag{10}$$

where

$$p_j = \frac{\partial S(t, q_i)}{\partial q^j}, \tag{11}$$

$$p_0 = \frac{\partial S(t, q_i)}{\partial t}, \tag{12}$$

and $i, j = 1, ..., N$, $\alpha = 1, ..., R$; $a = R + 1, ..., N$; H_c is the canonical Hamiltonian and S is the Hamilton principal function.

It can be shown that the equations of motion are total differential equations for the characteristics curves of the differential partial equations (9) and (10), being given by [5]

$$dq_i = \frac{\partial H'_0}{\partial p^i} dt + \frac{\partial H'_\alpha}{\partial p^i} dq_\alpha, \tag{13}$$

$$dp_i = -\frac{\partial H'_0}{\partial q^i} dt - \frac{\partial H'_\alpha}{\partial q^i} dq_\alpha, \tag{14}$$

where for $i = 1, ..., R$ equation (13) above becomes a trivial identity. Using standard techniques of partial differential equations, it can be shown [12] that the equations above are integrable if and only if the functions H'_α satisfy

$$dH'_\beta = \left\{H'_\beta, H'_\alpha\right\} dt^\alpha, \tag{15}$$

where $\alpha, \beta = 0, 1, ..., R$; $t_0 = t$ (so that $t_\nu = (t, q_\alpha)$) and the symbol $\{..., ...\}$ denotes the Poisson bracket defined on the phase space of $2N+2$ dimension that includes $t_0 = t$ and its canonical momentum p_0.

3.1 The Teleparallel Lagrangian case

Let us now consider the Lagrangian density given in Eq. (3) in the context of the Hamilton-Jacobi formalism. For this purpose we note that this Lagrangian density does not depend on time derivatives of the shift and lapso functions, therefore we define the set of Hamilton-Jacobi partial differential equations as

$$H'_{0c} = \int d^3x (H_c(x) + p_0(x)) = 0, \tag{16}$$

$$H'_0 = \int d^3x \Pi_0(x) = 0, \tag{17}$$

$$H'_i = \int d^3x \Pi_i(x) = 0, \quad i = 1, ..3; \tag{18}$$

where Π_0 and Π_i are the canonical momenta conjugated to the shift and lapso functions, respectively; H_c is the canonical Hamiltonian and p_0 is the "momentum" conjugated to the time parameter. So, in this approach, both N and N^i are taken as evolution parameters, together with t, from the beginning. Now we apply the integrability conditions given by Eq. (15) to the equations above. After some calculation we obtain

$$dH'_0 = \{H'_0, H'_{0c}\}\, dt = -\left(\int d^3x C(x)\right) dt = 0, \tag{19}$$

$$dH'_i = \{H'_i, H'_{0c}\}\, dt = -\left(\int d^3x C_i(x)\right) dt = 0, \tag{20}$$

$$dH'_{0c} = \{H'_{0c}, H'_{0c}\}\, dt + \{H'_c, H'_0\}\, dN + \{H'_c, H'_i\}\, dN^i = \left[\int d^3x \frac{\partial}{\partial x^i}\left(F^i(x)\right)\right] dt = 0, \tag{21}$$

where

$$F^i = N^2\left(-\delta^i_j \partial_m - \frac{1}{2} T^i{}_{mj} + T_{mj}{}^i\right) \Pi^{[mj]}, \quad m, j = 1, ..., 3; \tag{22}$$

with the brackets on the indexes indicating antisymmetrization. Equation (21) above can be transformed in a surface integral and, therefore, at the surface of integration we must have

$$F^i = N^2 \left(-\delta^i_j \partial_m - \frac{1}{2} T^i{}_{mj} + T_{mj}{}^i \right) \Pi^{[mj]} = 0. \tag{23}$$

However, the latest equation is a consequence of the fact that the canonical momenta Π^{mj} in this approach are symmetrical, so the antisymmetric components $\Pi^{[mj]}$ must vanish, therefore the Equation (23) is satisfied in in the whole spacetime, what corresponds to primary constraints in the approach of reference [14].

The integrability conditions given by equations (19), (20) and (21) imply that $C = 0$, $C_i = 0$ and $\Pi^{[mj]} = 0$ and no new conditions arise. The integrability conditions are equivalent to the consistency conditions obtained in reference [14] using Dirac's method, and the quantities C, C_i, $\Pi^{[mj]}$ constitute a set of first class constraints.

4 Final remarks

In this work we analyzed the Lagrangian density of TEGR, with a specific choice of reference frame, by using the Hamilton-Jacobi formalism, which was recently developed to treat singular systems. Such Lagrangian has already been studied through Dirac's Hamiltonian formalism, where the consistency conditions produce a set of first class constraints [14]. In our analysis, the integrability conditions of Hamilton-Jacobi formalism produce results that are identical to those obtained in reference [14] through the use Dirac's Hamiltonian method.

However, one of the most interesting characteristics of Hamilton-Jacobi formalism is the possibility of avoiding specific choices of gauge or reference systems. So, we believe to be possible to study the Lagrangian density of TEGR without assuming any *a priori*

restriction on the tetrad fields, like Schwinger's time gauge. Our expectation is that such restrictions should naturally arise as consequence of integrability conditions in Hamilton-Jacobi formalism, as happens in other singular systems [29, 30]. This question is presently under our study.

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